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Faxen's laws for a micropolar fluid

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Summary

Faxen's formulas for the drag and torque on a rigid spherical particle immersed in a Stokes flow of a viscous incompressible fluid are extended for the case of an incompressible micropolar fluid.

1. Introduction

Faxen's laws [1] (see also Brenner [2]) for the drag F_i and torque T_i exerted on a rigid stationary spherical particle of radius a immersed in an arbitrary Stokes flow field, with velocity vector $u_i = u_i(x_1, x_2, x_3)$, extending to infinity are

$$F_i = 6\pi\mu a\left[(u_i)_0 + \tfrac{1}{6}a^2(\nabla^2 u_i)_0\right], \tag{1.1}$$

$$T_i = 4\pi\mu a^3\left[(\epsilon_{ijk}u_{k,j})_0\right]. \tag{1.2}$$

where the subscript zero indicates the evaluation at the centre of the sphere. In this paper, these laws are extended for the case of a homogeneous incompressible micropolar fluid. In the absence of inertial effects, body forces and body couples, the equations of motion for a homogeneous incompressible micropolar fluid are [3],

$$u_{i,i} = 0, \tag{1.3}$$

$$(\mu+\kappa)u_{i,jj} + \kappa\epsilon_{ijk}\nu_{k,j} - p_{,i} = 0, \tag{1.4}$$

$$(\alpha+\beta)\nu_{j,ij} + \gamma\nu_{i,jj} + \kappa\epsilon_{ijk}u_{k,j} - 2\kappa\nu_i = 0. \tag{1.5}$$

Here ν_i is the micro-rotation vector, p denotes the pressure and μ, κ, α, β, γ the material constants of the micropolar fluid; ϵ_{ijk} is the alternating tensor. A comma denotes partial differentiation and a repeated index implies a summation over the three possible values 1, 2, 3 of the index.

The constitutive equations for the stress tensor σ_{ij} and couple stress tensor m_{ij} are

$$\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + \kappa(u_{j,i} - \epsilon_{ijk}\nu_k), \tag{1.6}$$

$$m_{ij} = \alpha\nu_{k,k}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}. \tag{1.7}$$

The material constants in (1.3)–(1.7) are restricted by the Clausius-Duhem inequalities

$$(2\mu+\kappa)\geqslant 0, \qquad \kappa\geqslant 0,$$

$$(3\alpha+\beta+\gamma)\geqslant 0, \qquad \gamma\geqslant|\beta|. \tag{1.8}$$

2. The generalized reciprocal theorem

The reciprocal theorem which was originally given by Brenner [2] for the case of a classical viscous fluid has recently been extended by Ramkissoon et al. [4,5] for a micropolar fluid. This theorem is recalled here as it is needed in the subsequent derivation.

Theorem:
Let (u', ν', p', σ'_{ij}, m'_{ij}) and (u'', ν'', p'', σ''_{ij}, m''_{ij}) represent any two motions of the same micropolar fluid which conform to equations (1.3)–(1.7). Let $\partial\Omega$ be a closed surface bounding any fluid volume Ω and u', ν', u'', $\nu''\in\mathscr{C}^1$ in $\partial\Omega+\Omega$. Then we have the following reciprocal relationship,

$$\int_{\partial\Omega}(n_j\sigma'_{jk}u''_k+n_jm'_{jk}\nu''_k)\mathrm{d}S=\int_{\partial\Omega}(n_j\sigma''_{jk}u'_k+n_jm''_{jk}\nu'_k)\mathrm{d}S, \tag{2.1}$$

it being assumed that the fields (u'_k, ν'_k) and (u''_k, ν''_k) vanish at infinity.

3. Drag on an arbitrary particle

Consider the motion of a particle S of any shape in a homogenous incompressible micropolar fluid which is at rest at infinity.

Let (u'_k, ν'_k) be the solution of the field equations (1.3)–(1.5) satisfying the boundary conditions

$$u'_k=U'_k, \qquad \nu'_k=0 \qquad \text{on } S; \tag{3.1}$$

$$u'_k\to 0, \qquad \nu'_k\to 0 \qquad \text{as } r\to\infty, \tag{3.2}$$

where the constant vector U'_k is arbitrary. Further, owing to the linearity of the equations of motion and the boundary conditions, the stress tensor σ'_{jk} and the couple stress tensor m'_{jk} may be expressed as (see Brenner [6]),

$$\sigma'_{jk}=(2\mu+\kappa)L_{jk\ell}U'_\ell, \tag{3.3}$$

$$m'_{jk}=(2\mu+\kappa)M_{jk\ell}U'_\ell, \tag{3.4}$$

where $L_{jk\ell}$ and $M_{jk\ell}$ are third-order tensors depending on the shape of the particle.

Now let (u''_k, ν''_k) be any solution of the field equations (1.3)–(1.5) satisfying arbitrary boundary conditions on S and vanishing at infinity. The drag force experienced by the particle S due to the field variables (u_k'', ν_k'') is

$$F_k'' = \int_S n_j \sigma_{jk}'' \mathrm{d}S. \tag{3.5}$$

The scalar product of Eqn. (3.5) with the vector U_k' yields

$$F_k'' U_k' = \int_S (n_j \sigma_{jk}'' u_k' + n_j m_{jk}'' \nu_k') \mathrm{d}S, \tag{3.6}$$

by virtue of (3.1). Now the use of the generalized reciprocal theorem (2.1) and the relations (3.3) and (3.4) together with the fact that U_k' is an arbitrary constant vector gives

$$F_k'' = (2\mu + \kappa) \int_S (n_j L_{j\ell k} u_\ell'' + n_j M_{j\ell k} \nu_\ell'') \mathrm{d}S. \tag{3.7}$$

The equation (3.7) gives the drag due to the flow field (u_k'', ν_k'') which vanishes at infinity. To remove this restriction, let (u_k, ν_k) be the solution of field equations (1.3)–(1.5), satisfying arbitrary conditions on the surface of the particle and tending to a prescribed Stokes flow (u_k^*, ν_k^*) at infinity. The fields $u_k'' = u_k - u_k^*$, $\nu_k'' = \nu_k - \nu_k^*$ then satisfy the equations (1.3)–(1.5) and vanish at infinity. Since by linearity, $F_k'' = F_k - F_k^*$ and the field (u_k^*, ν_k^*) is free from singularities in the interior of the space occupied by the particle and cannot produce any force on the particle, it follows that $F_k'' = F_k$. Therefore

$$F_k = (2\mu + \kappa) \int_S \left[n_j L_{j\ell k} (u_\ell - u_\ell^*) + n_j M_{j\ell k} (\nu_\ell - \nu_\ell^*) \right] \mathrm{d}S. \tag{3.8}$$

The equation (3.8) gives the drag on the particle which is immersed in an arbitrary Stokes flow (u_ℓ^*, ν_ℓ^*) at infinity and which satisfies arbitrary conditions on the surface S. The drag on the particle which is maintained at rest in the flow (u_ℓ^*, ν_ℓ^*) is obtained by putting $u_\ell = 0$, $\nu_\ell = 0$ in (3.8). Thus,

$$F_k = -(2\mu + \kappa) \int_S (n_j L_{j\ell k} u_\ell^* + n_j M_{j\ell k} \nu_\ell^*) \mathrm{d}S. \tag{3.9}$$

4. Drag on a sphere

Consider a spherical particle S of radius a with the origin at the centre of the sphere. From the solutions given by Lakshmana Rao et al. [7] for the uniform motion of a sphere, we find that

$$n_j L_{j\ell k} = -\frac{3(al + 1)(\mu + \kappa)}{2a[2(\mu + \kappa)al + 2\mu + \kappa]} \delta_{\ell k}, \tag{4.1}$$

$$n_j M_{j\ell k} = \frac{3\kappa}{2a[2(\mu + \kappa)al + 2\mu + \kappa]} \epsilon_{\ell k m} x_m, \tag{4.2}$$

where

$$l^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}.$$

On using (4.1) and (4.2) in (3.9), we get

$$F_k = \frac{3(2\mu+\kappa)}{2a[2(\mu+\kappa)al+2\mu+\kappa]}\left[(al+1)(\mu+\kappa)\int_S u_k^* \mathrm{d}S - \kappa\int_S \epsilon_{km\ell}x_m v_\ell^* \mathrm{d}S\right]. \tag{4.3}$$

Now it is easy to show that, for any vector functions u_i, v_i possessing continuous derivatives at the origin, the following identities for the surface integrals on the sphere hold:

$$\int_S u_i \mathrm{d}S = 4\pi a^2\left[(u_i)_0 + \frac{a^2}{3!}(\nabla^2 u_i)_0 + \frac{a^4}{5!}(\nabla^4 u_i)_0 + \dots\right], \tag{4.4}$$

$$\int_S \epsilon_{ijk}x_j v_k \mathrm{d}S = 4\pi a^4\left[\frac{2}{3!}(\epsilon_{ijk}v_{k,j})_0 + \frac{4a^2}{5!}\left(\nabla^2(\epsilon_{ijk}v_{k,j})\right)_0 + \frac{6a^4}{7!}\left(\nabla^4(\epsilon_{ijk}v_{k,j})\right)_0 + \dots\right]. \tag{4.5}$$

The suffix zero indicates that all the functions evaluated at the centre of the sphere. Using (4.4) and (4.5) in (4.3) gives the Faxen law for the drag:

$$F_k = \frac{6\pi a(2\mu+\kappa)}{[2(\mu+\kappa)al+2\mu+\kappa]}\Bigg\{(al+1)(\mu+\kappa)\left[(u_k^*)_0 + \frac{a^2}{3!}(\nabla^2 u_k^*)_0 + \frac{a^4}{5!}(\nabla^4 u_k^*)_0 + \dots\right] - \kappa a^2\left[\frac{2}{3!}(\epsilon_{km\ell}v_{\ell,m}^*)_0 + \frac{4a^2}{5!}\left(\nabla^2(\epsilon_{km\ell}v_{\ell,m}^*)\right)_0 + \frac{6a^4}{7!}\left(\nabla^4(\epsilon_{km\ell}v_{\ell,m}^*)\right)_0 + \dots\right]\Bigg\}. \tag{4.6}$$

This formula reduces to the classical Faxen law (1.1) when the material constant $\kappa = 0$, since $\nabla^4 u_k^* = \nabla^6 u_k^* = \dots = 0$ for a classical viscous fluid.

Examples

(i) Uniform flow past a sphere
The undisturbed Stokes flow field is given by $u_k^* = (U, 0, 0)$ and $v_\ell^* = (0, 0, 0)$, the Faxen law (4.6) readily gives

$$F_1 = \frac{6\pi aU(2\mu+\kappa)(al+1)(\mu+\kappa)}{[2(\mu+\kappa)al+2\mu+\kappa]}, \tag{4.7}$$

$$F_2 = F_3 = 0, \tag{4.8}$$

which agrees with the result of Lakshmana Rao [7].

(ii) Shear flow past a sphere
Let the undisturbed flow field be given by $u_k^* = a_{k\ell} x_\ell$, $\nu_k^* = \frac{1}{2}\epsilon_{km\ell} a_{\ell m}$, then the Faxen law (4.6) shows

$$F_k = (0, 0, 0) \tag{4.9}$$

which agrees with the result of Niefer and Kaloni [8].

5. Torque on a sphere

Let (u_k', ν_k') be the solution of the field equations (1.3)–(1.5) satisfying the boundary conditions

$$u_k' = \epsilon_{k\ell m}\omega_\ell x_m, \qquad \nu_k' = \phi_k \qquad \text{on } S; \tag{5.1}$$

$$u_k' \to 0, \qquad \nu_k' \to 0 \qquad \text{as } r \to \infty, \tag{5.2}$$

where ω_ℓ and ϕ_k are arbitrary constant vectors. By linearity, the stress tensor σ_{jk}' and the couple stress tensor m_{jk}' can be written as

$$\sigma_{jk}' = (2\mu + \kappa)\left[P_{jk\ell}\omega_\ell + P_{jk\ell}^*\phi_\ell\right] \tag{5.3}$$

$$m_{jk}' = (2\mu + \kappa)\left[Q_{jk\ell}\omega_\ell + Q_{jk\ell}^*\phi_\ell\right] \tag{5.4}$$

where $P_{jk\ell}$, $P_{jk\ell}^*$, $Q_{jk\ell}$, $Q_{jk\ell}^*$ are third-order tensors depending on the shape of the particle.

Now, let (u_k'', ν_k'') be the solution of the field equations (1.3)–(1.5) satisfying arbitrary boundary conditions on the surface of the particle and vanishing at infinity. The torque on the particle M_ℓ'' due to the stress tensor σ_{jk}'' is given by

$$M_\ell'' = \int_S \epsilon_{\ell mk} x_m \sigma_{jk}'' n_j \, \mathrm{d}S. \tag{5.5}$$

The torque N_ℓ'' due to the couple stress $m_{j\ell}''$ is given by

$$N_\ell'' = \int_S m_{j\ell}'' n_j \, \mathrm{d}S. \tag{5.6}$$

Taking the scalar products of the equations (5.5) and (5.6) with the vectors ω_ℓ and ϕ_ℓ respectively and adding, we get the following equation by virtue of (5.1)

$$M_\ell''\omega_\ell + N_\ell''\phi_\ell = \int_S \left(\sigma_{jk}'' n_j u_k' + m_{jk}'' n_j \nu_k'\right)\mathrm{d}S. \tag{5.7}$$

Now the use of the generalized reciprocal theorem (2.1) gives

$$M_\ell''\omega_\ell + N_\ell''\phi_\ell = \int_S \left(\sigma_{jk}' n_j u_k'' + m_{jk}' n_j \nu_k''\right)\mathrm{d}S. \tag{5.8}$$

On using (5.3) and (5.4) in (5.8), we get

$$M_\ell''\omega_\ell + N_\ell''\phi_\ell = (2\mu+\kappa)\int_S\Big[\big(P_{jk\ell}n_ju_k'' + Q_{jk\ell}n_jv_k''\big)\omega_\ell + \big(P_{jk\ell}^*n_ju_k'' + Q_{jk\ell}^*n_jv_k''\big)\phi_\ell\Big]\mathrm{d}S. \tag{5.9}$$

The equation (5.9) is true for any arbitrary constant vectors ω_ℓ and ϕ_ℓ. Considering the case when $\phi_\ell \equiv 0$ and $\omega_\ell \neq 0$, we have

$$M_\ell'' = (2\mu+\kappa)\int_S\big(P_{jk\ell}n_ju_k'' + Q_{jk\ell}n_jv_k''\big)\mathrm{d}S. \tag{5.10}$$

Similarly the case $\omega_\ell \equiv 0$ and $\phi_\ell \neq 0$ gives

$$N_\ell'' = (2\mu+\kappa)\int_S\big(P_{jk\ell}^*n_ju_k'' + Q_{jk\ell}^*n_jv_k''\big)\mathrm{d}S. \tag{5.11}$$

Therefore, the total torque $T_\ell'' = M_\ell'' + N_\ell''$ is given by

$$T_\ell'' = (2\mu+\kappa)\int_S\big(A_{jk\ell}n_ju_k'' + B_{jk\ell}n_jv_k''\big)\mathrm{d}S \tag{5.12}$$

where $A_{jk\ell} = P_{jk\ell} + P_{jk\ell}^*$; $B_{jk\ell} = Q_{jk\ell} + Q_{jk\ell}^*$. The equation (5.12) gives the torque due to the flow field (u_k'', v_k'') which vanishes at infinity. To remove this restriction, we again assume $u_k'' = u_k - u_k^*$, $v_k'' = v_k - v_k^*$. Then we have (with arguments similar to those used in Section 3):

$$T_\ell = (2\mu+\kappa)\int_S\big[n_jA_{jk\ell}(u_k - u_k^*) + n_jB_{jk\ell}(v_k - v_k^*)\big]\mathrm{d}s. \tag{5.13}$$

The equation (5.13) gives the torque on the particle which is immersed in an arbitrary Stokes flow (u_k^*, v_k^*) at infinity and which satisfies arbitrary conditions on the surface of the particle. The torque on a particle which is maintained at rest in the flow (u_k^*, v_k^*) is obtained by putting $u_\ell = 0$, $v_\ell = 0$ in (5.13). Thus,

$$T_\ell = -(2\mu+\kappa)\int_S\big(n_jA_{jk\ell}u_k^* + n_jB_{jk\ell}v_k^*\big)\mathrm{d}S. \tag{5.14}$$

Now, from the solution given by Lakshmana Rao et al. ([9], Eqn.48) for the slow steady rotation of a sphere, we find that

$$n_jA_{jk\ell} = -\frac{3}{aD}(\mu+\kappa)\big[(c^2+2c+2)a^2l^2 + c^2(1+al)\big]\epsilon_{k\ell m}x_m \tag{5.15}$$

where

$$D = 2(\mu+\kappa)(c^2+2c+2)a^2l^2 + c^2(2\mu+\kappa)(1+al)$$

and

$$\frac{c^2}{a^2} = \frac{2\kappa}{\alpha+\beta+\gamma}.$$

Along similar lines it can be shown that the tensor $B_{jk\ell}$ for the sphere is given by

$$n_j B_{jk\ell} = -\frac{2a}{D}\kappa(1+al)(c^2+3c+3)\delta_{k\ell}. \tag{5.16}$$

Therefore, the torque on the sphere is given by

$$T_\ell = \frac{(2\mu+\kappa)}{D}\left\{\frac{3}{a}(\mu+\kappa)\left[(c^2+2c+2)a^2l^2 + c^2(1+al)\right]\right.$$

$$\left.\times \int_S \epsilon_{\ell mk} x_m u_k^* \mathrm{d}S + 2a\kappa(1+al)(c^2+3c+3)\int_S v_\ell^* \mathrm{d}S\right\}. \tag{5.17}$$

On using the identities (4.4) and (4.5) in (5.17), we get the Faxen law for the torque on the sphere,

$$T_\ell = \frac{12\pi a^3(2\mu+\kappa)}{D}\left\{(\mu+\kappa)\left[(c^2+2c+2)a^2l^2 + c^2(1+al)\right]\right.$$

$$\times\left[\frac{2}{3!}(\epsilon_{\ell mk}u^*_{k,m})_0 + \frac{4a^2}{5!}\left(\nabla^2(\epsilon_{\ell mk}u^*_{k,m})\right)_0 + \ldots\right]$$

$$+\tfrac{2}{3}\kappa(1+al)(c^2+3c+3)$$

$$\left.\times\left[(v_\ell^*)_0 + \frac{a^2}{3!}(\nabla^2 v_\ell^*)_0 + \frac{a^4}{5!}(\nabla^4 v_\ell^*)_0 + \ldots\right]\right\} \tag{5.18}$$

where the suffix zero indicates that all the functions are evaluated at the centre of the sphere. The classical Faxen law (1.2) can be recovered in the limit $\kappa \to 0$, $\gamma \to 0$. In this case $l \to 0$, $c \to 0$ as $\kappa \to 0$

$$\frac{(\mu+\kappa)\left[(c^2+2c+2)a^2l^2 + c^2(1+al)\right]}{D} \to \frac{1}{2},$$

$$\frac{\kappa(1+al)(c^2+3c+3)}{D} \to 0,$$

and (5.18) reduces to (1.2), since $\nabla^2(\epsilon_{\ell mk}u^*_{k,m}) = \nabla^4(\epsilon_{\ell mk}u^*_{k,m}) = \ldots = 0$ for a classical viscous fluid.

Examples

1. Sphere in a rotating fluid
The undisturbed flow field is given by $u_k^* = (-\omega x_2, \omega x_1, 0)$ and $\nu_\ell^* = (0, 0, \omega)$, then the Faxen law (5.18) gives

$$T_1 = T_2 = 0, \tag{5.19}$$

$$T_3 = \frac{8\pi\omega a^3(2\mu + \kappa)}{D}\left\{(\mu + \kappa)\left[(c^2 + 2c + 2)a^2 l^2 + c^2(1 + al)\right]\right.$$

$$\left. + \kappa(1 + al)(c^2 + 3c + 3)\right\}. \tag{5.20}$$

2. Rotational shear at infinity
The undisturbed flow field is given by $u_k^* = a_{k\ell} x_\ell$, $\nu_\ell^* = \frac{1}{2}\epsilon_{\ell mk} a_{km}$, then Faxen's law (5.18) gives

$$T_\ell = \frac{4\pi a^3(2\mu + \kappa)}{D}\left\{(\mu + \kappa)\left[(c^2 + 2c + 2)a^2 l^2 + c^2(1 + al)\right]\right.$$

$$\left. + \kappa(1 + al)(c^2 + 3c + 3)\right\}\epsilon_{\ell mk} a_{km}. \tag{5.21}$$

If a_{km} is symmetric, then the torque becomes zero,

$$T_\ell = (0, 0, 0). \tag{5.22}$$

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